## SECTION C

### NUMERICAL METHODS 1 (3.09) taught by David Ham

Candidates being examined in "Numerical Methods 1" should answer at least one question from Section C.

C1. (i) Consider the number 5.25

(a) Write the number in the form:

$$s1.m \times 2^{e-b} \tag{1}$$

Use a floating point format with 3 exponent bits and 4 mantissa bits, and a bias of 3. All the numbers must be written in binary.

*(6 marks)*

(b) Convert this number into the bit pattern which would actually be stored, according to the layout provided on the formula sheet.

*(2 marks)*

(ii) (a) The Bisection method is given by the algorithm:

$f_l \leftarrow f(x_l)$
**repeat**
  $x_c \leftarrow \frac{x_l + x_r}{2}$
  $f_c \leftarrow f(x_c)$
  **if** $f_c f_l > 0$ **then**
    $x_l \leftarrow x_c$
    $f_l \leftarrow f_c$
  **else**
    $x_r \leftarrow x_c$
**until** $|f(x_c)| < \epsilon$

Produce a sketch of one iteration of this algorithm for the function $f(x) = x^2 - 1$ with a starting interval $x_l = 0.5, x_r = 2$. Show all the relevant points and lines and indicate which end of the interval will be removed.

*(3 marks)*

(b) The Newton-Raphson iteration is given by the algorithm:

**repeat**
  $f_x \leftarrow f(x)$
  $x \leftarrow x - \frac{f_x}{f'(x)}$
**until** $(|f_x| < \epsilon)$ or maximum iterations exceeded.

Produce a sketch of one iteration of this algorithm for the function $f(x) = x^2 - 1$ with a starting position $x_0 = 0.5$. Show all the relevant points and lines.

*(3 marks)*

(c) What is the key advantage of Newton-Raphson iteration over the bisection method?

*(1 mark)*

(d) What additional information is required to use Newton-Raphson iteration rather than the bisection method?

*(1 mark)*

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(e) What advantage does the bisection method have over Newton-Raphson iteration?

*(1 mark)*

(iii) Suppose that A is a rectangular $n \times m$ matrix with $n > m$, and that the columns of A are all orthogonal to each other, and each column has a modulus of 1 ($|\mathrm{A}_{:,i}| = \sqrt{A_{:,i} \cdot \mathrm{A}_{:,i}} = 1$ for $0 \leq i \leq m-1$).

(a) prove that $\mathrm{A}^{\mathrm{T}}\mathrm{A} = \mathrm{I}$ where I is the $m \times m$ identity matrix.

*(4 marks)*

(b) prove therefore that the least squares solution to the overdetermined matrix equation:

$$\mathrm{A}\mathbf{x} = \mathbf{b}$$

is given by $\mathbf{x} = \mathrm{A}^{\mathrm{T}}\mathbf{b}$

*(4 marks)*

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C2. (i) Convert the numbers in the following problems into 4 bit two's compliment signed binary integers before performing the calculation and converting back into base 10.

(a) $-5+3$

*(4 marks)*

(b) $-1\times 4$

*(4 marks)*

(ii) For each of the following Python functions, write a mathematical expression using only matrices and vectors which performs the same operation. In each case, state whether each input and output is a matrix or vector.

(a)
```
def function_1(a,b,c):
    import numpy

    d=numpy.zeros(numpy.size(b))

    for j in range(len(d)):
        d[j]=numpy.dot(a[j,:],c)-b[j]

    return d
```

*(4 marks)*

(b)
```
def function_2(a):
    import numpy

    c=a.shape[1]
    b=numpy.zeros((c,c))

    for i in range(c):
        for j in range(c):
            b[i,j]=numpy.dot(a[:,i],a[:,j])

    return b
```

*(4 marks)*

(iii) A central difference approximation to the third derivative of a function $f(x)$ is given by:

$$f'''(x)=\frac{-f(x-2h)+2f(x-h)-2f(x+h)+2f(x+2h)}{2h^3}+\mathcal{O}(h^p)$$

Where $h$ is some small positive step size, and $p$ is the order of convergence. Using Taylor series for the function $f$, or otherwise, prove that the order of convergence, $p$, is equal to 2.

You may use without proof any of the properties of $\mathcal{O}$.

*(9 marks)*

**END OF SECTION C**